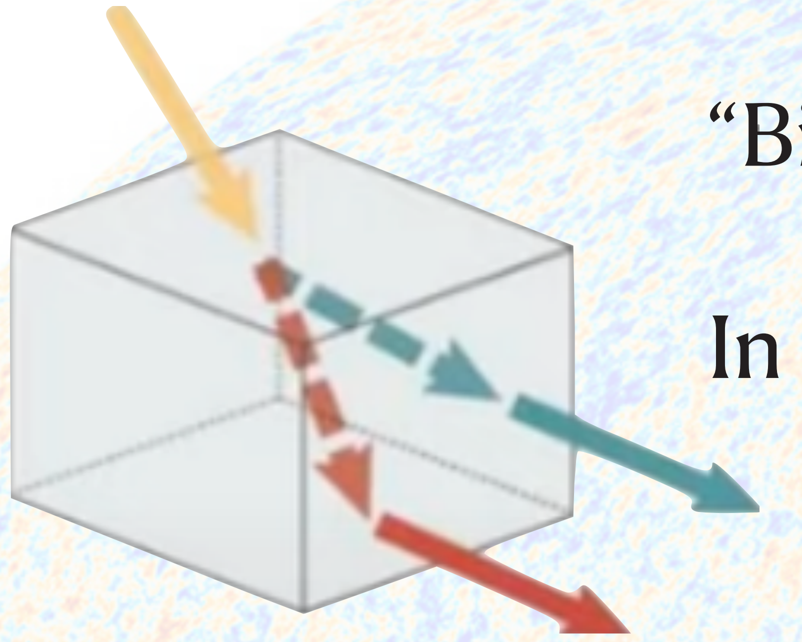
Workshop "CMB Day 2", ASI, 16-18 October 2023

Cosmic Birefringence:
How Our Universe May Violate Left-Right Symmetry

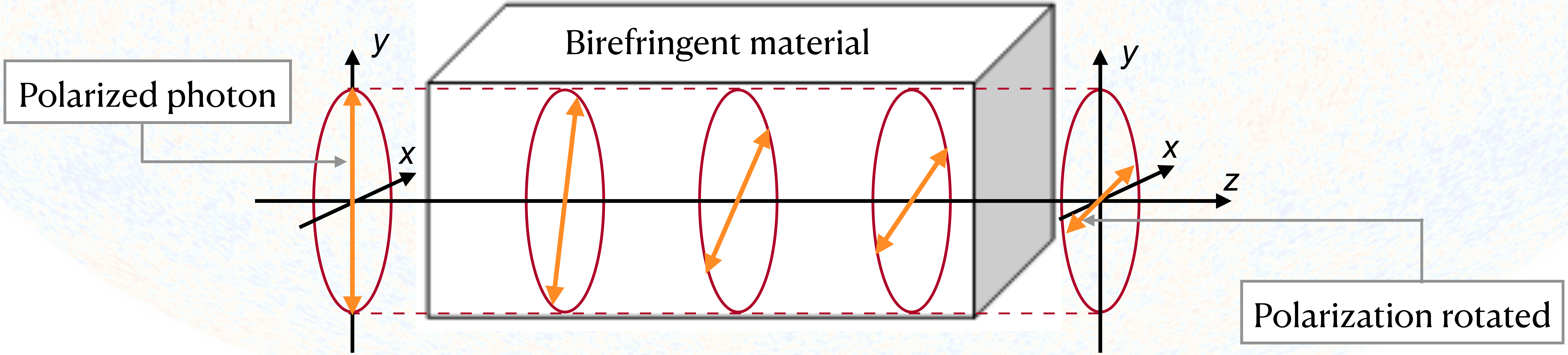
Margherita Lembo
University of Ferrara

What is (Cosmic) Birefringence?



“Birefringence” refers generically to the fact that wave normal modes propagate at different velocities.

In the **cosmological literature**, the term “cosmic birefringence” describes the specific case of different propagation velocity of circular polarization states (**rotation of the linear polarization plane**).



Theoretical Background

If the Universe is filled with a **parity-violating** pseudoscalar field (**axion-like particles, ALP**)

$$\phi(-\vec{n}) = -\phi(\vec{n}),$$

coupled to the electromagnetic tensor via a **Chern-Simons coupling**

$$\frac{1}{4} g_\phi \phi F_{\mu\nu} \tilde{F}^{\mu\nu},$$

it makes the phase velocities right- and left-handed helicity states of photons differ.

↓

Dispersion relation of left/right polarization are modified $\omega^2_{L/R} = k^2 \left[1 \pm \frac{g}{k} \dot{\phi} \right]$

Ni (1977), Turner&Widrow (1988), Carroll,Field,Jackiw(1990), Carroll&Field(1991).....

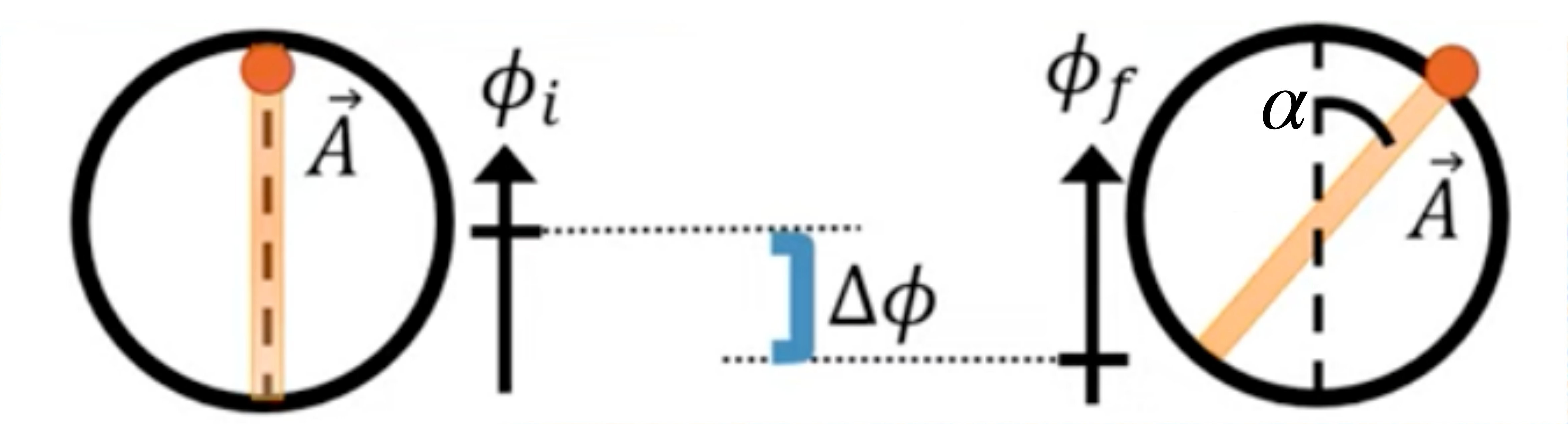
Theoretical Background

This results in a **rotation of linear polarization** plane by an angle

$$\alpha(\vec{n}) = -\frac{g_\phi}{2}\int \mathrm{d}t \frac{\partial \phi}{\partial t} = \frac{g_\phi}{2}\Delta\phi$$

↓

ALP causes Cosmic Birefringence



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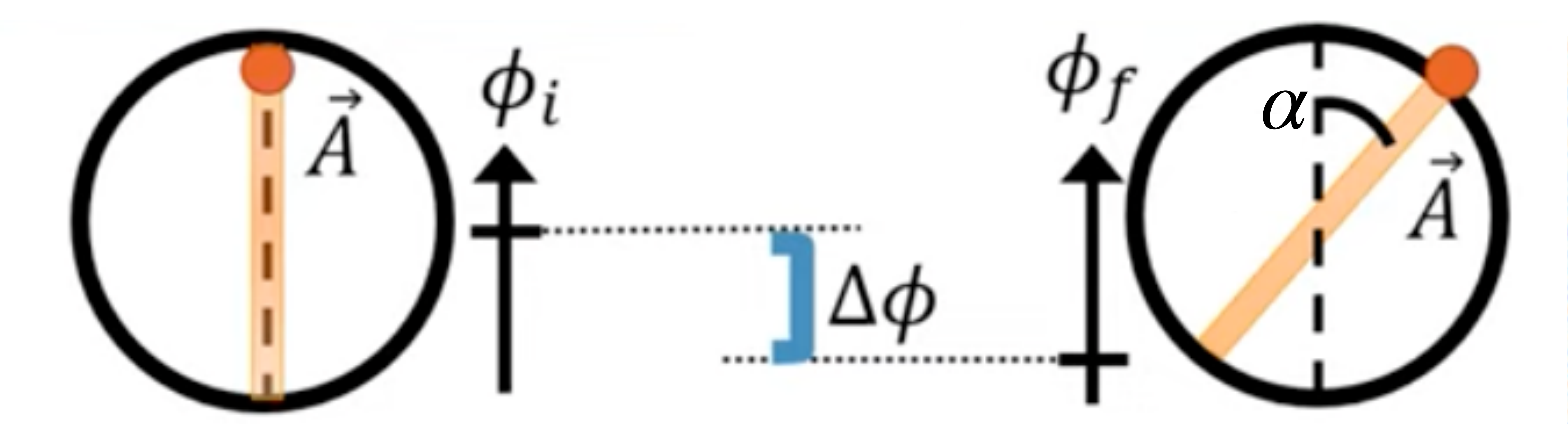

We can constrain ALP with a source that is:

- ☑ well-known
- ☑ linearly polarized
- ☑ very far away

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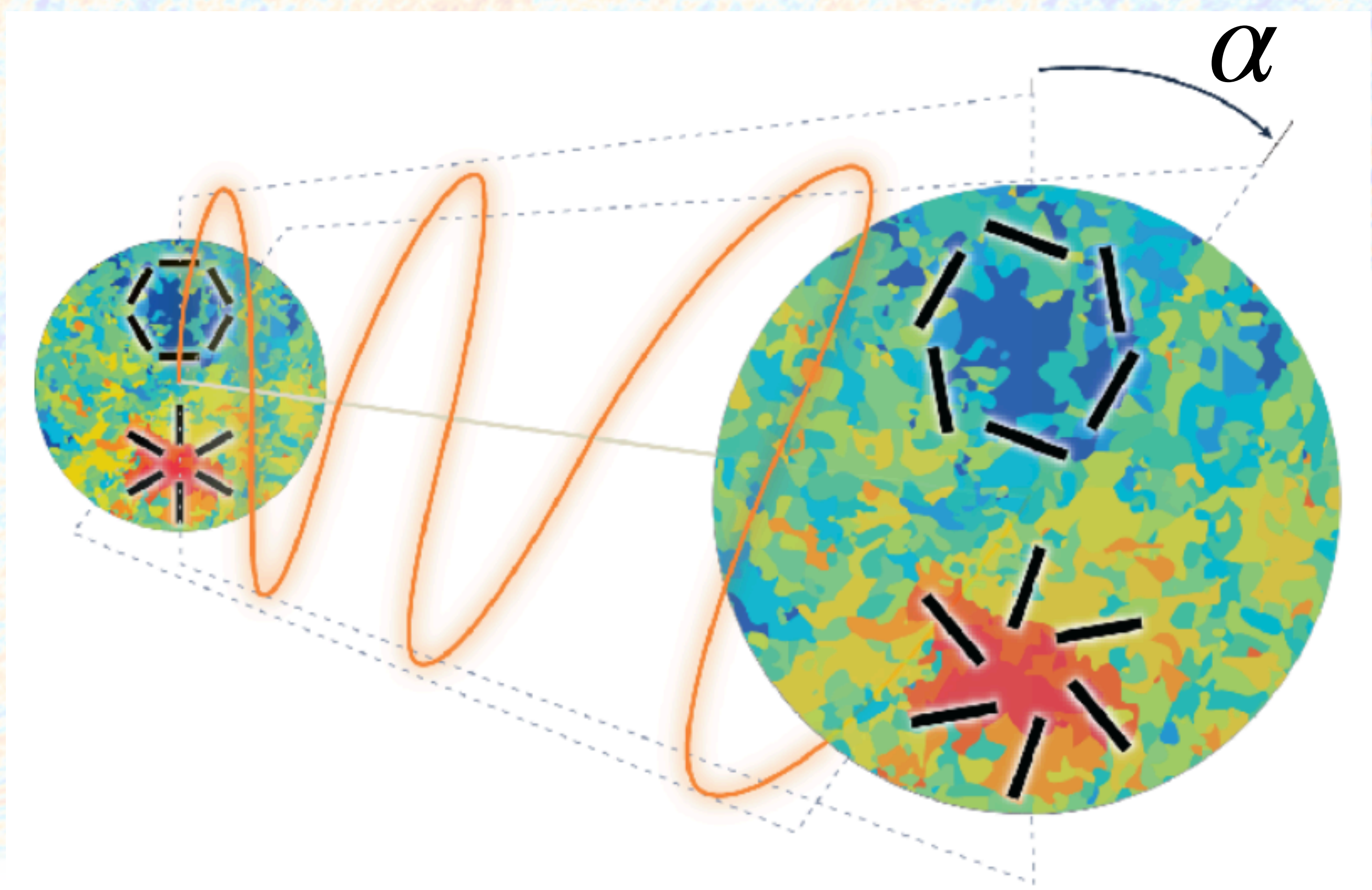
CMB



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Motivations

Axions motivated by
strong CP problem

↓

Peccei-Quinn
theory in 1977

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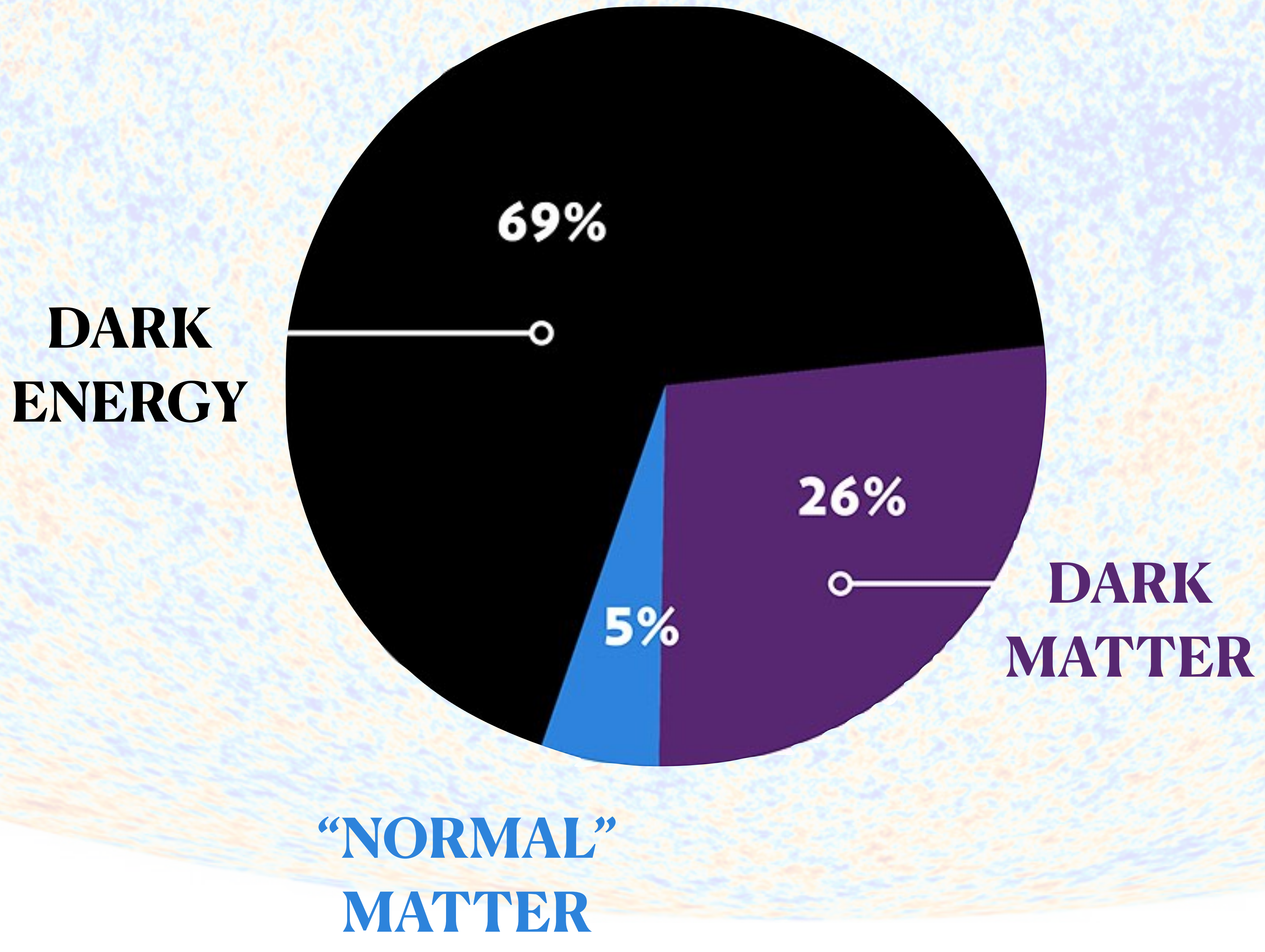
Peccei-Quinn theory in 1977

The dark side of the Universe is *obscure*

↓



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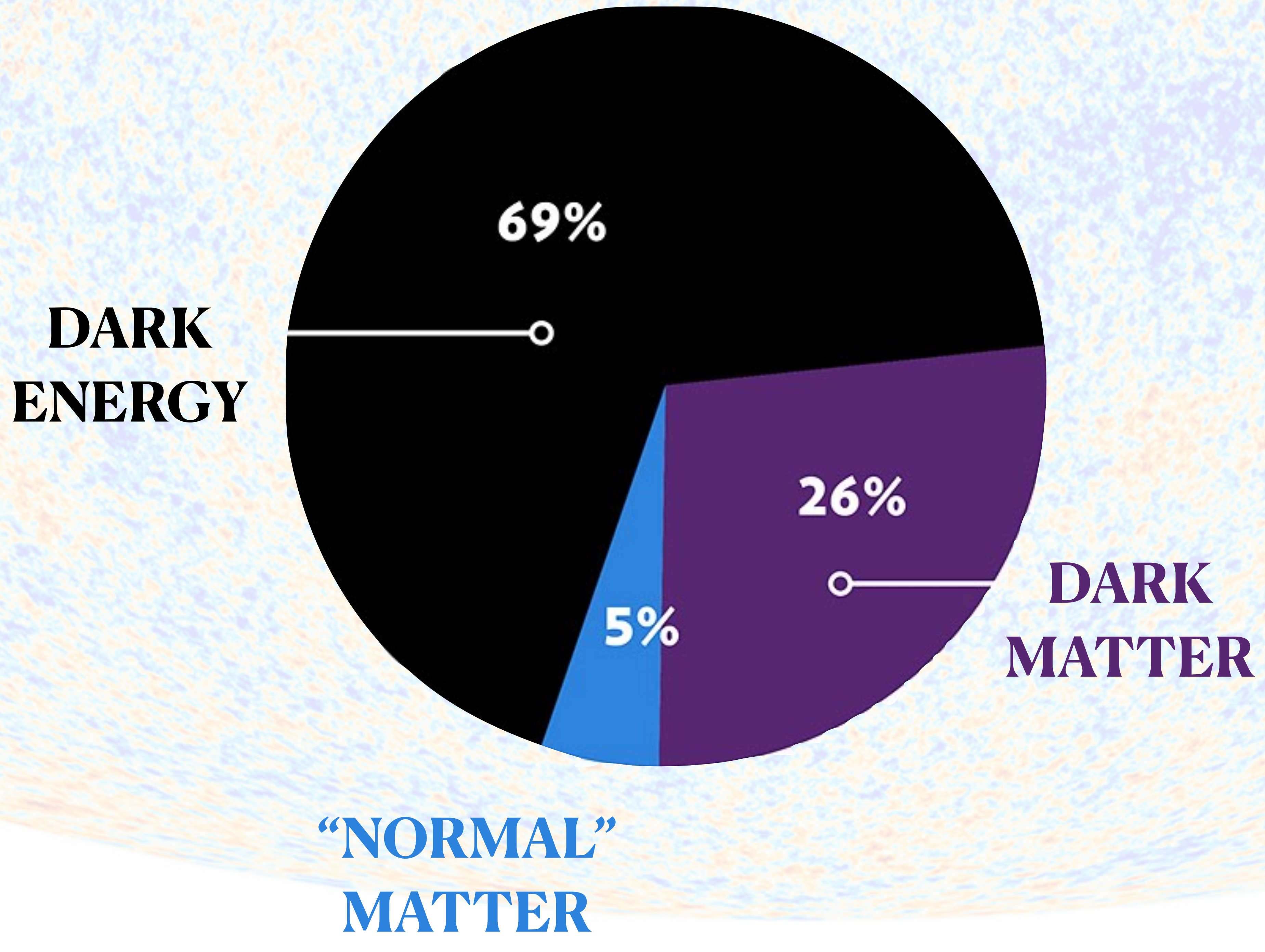
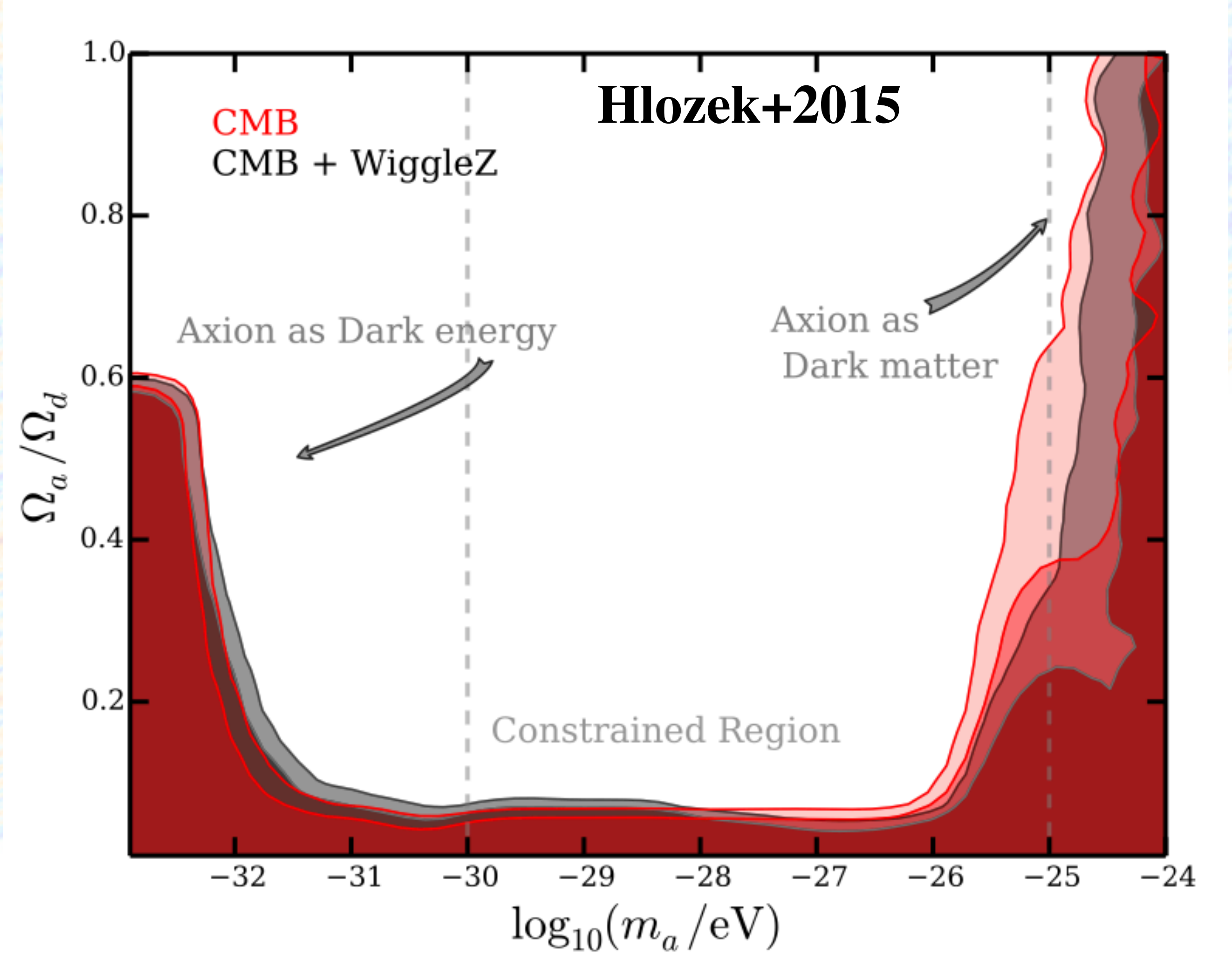


ALPs motivated by *String Axiverse*

↓

ALPs as **Dark Energy**: $m_\phi \lesssim H_0 \sim 10^{-33}$ eV

ALPs as **Dark Matter**: $m_\phi \gtrsim 10^{-27}$ eV



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Bonus Motivations

Parity symmetry is often broken:

- ✓ weak nuclear force
- ✓ DNA
- ✓ Right(left)-handed people
- ✓ ...

Why this symmetry should be preserved at *largest* scales?

ARK
TTER

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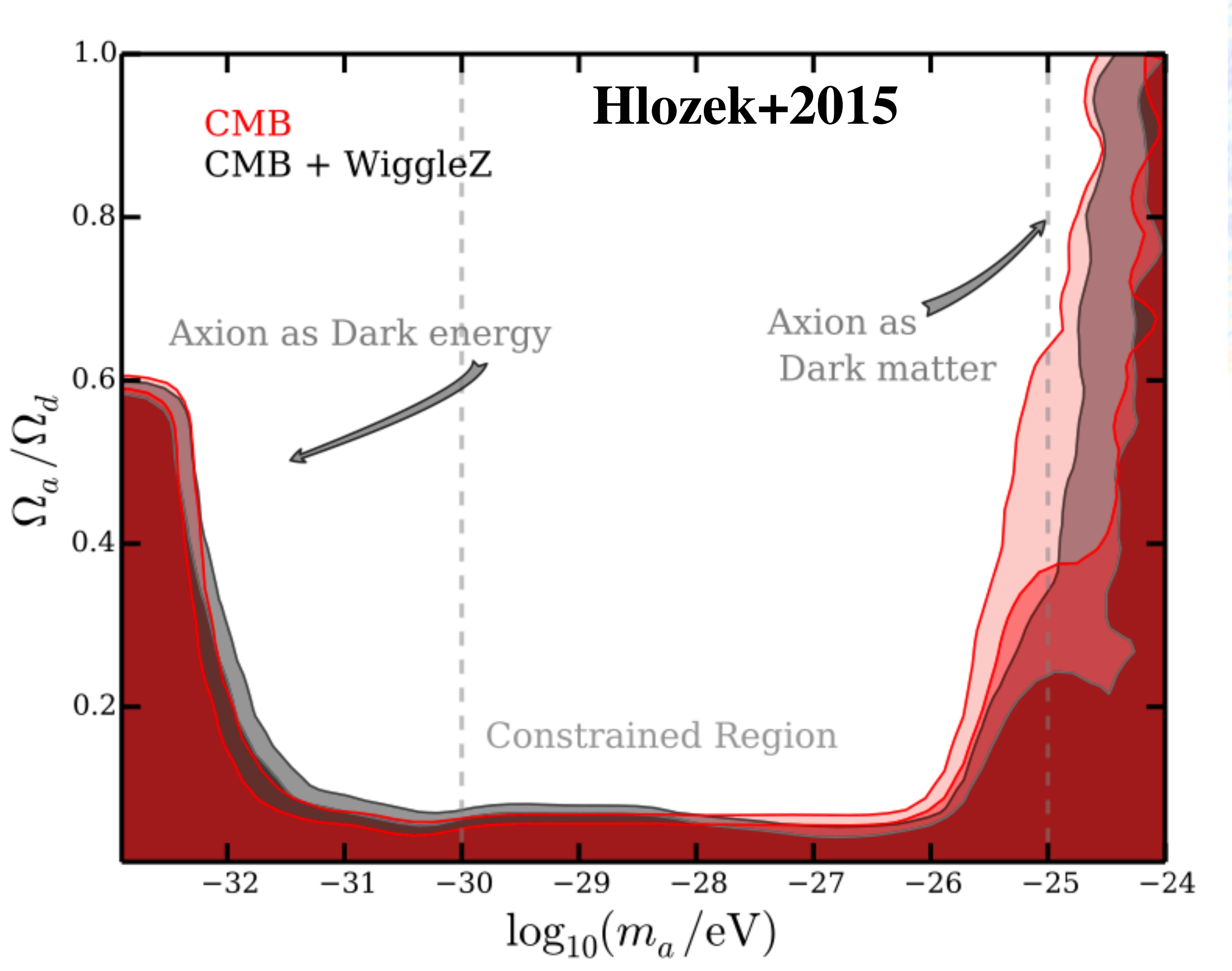

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Uni...

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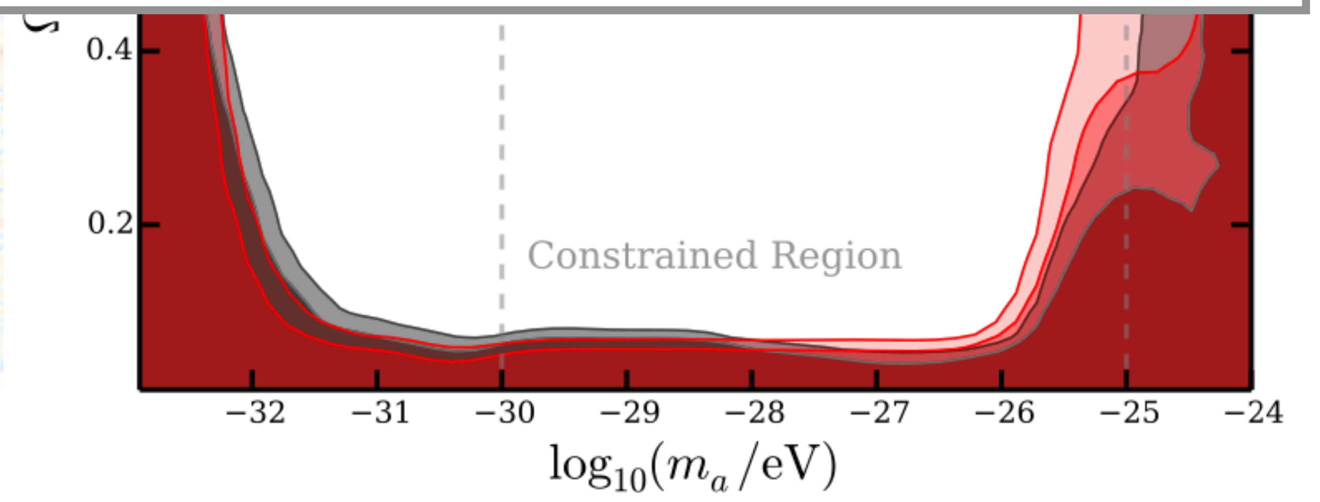
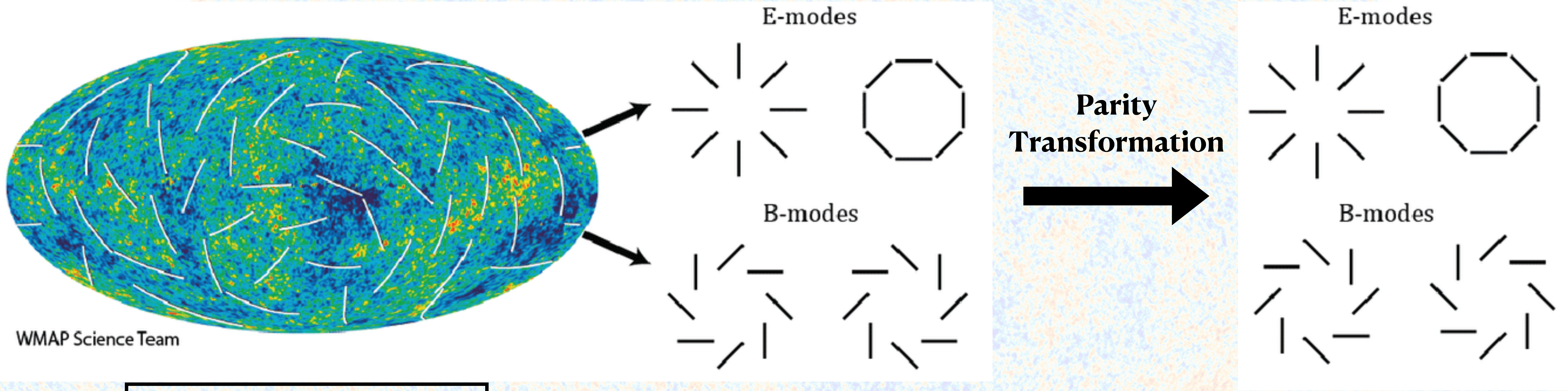
Bonus Motivations - Why "today"?

The precise measurement over a wide range of scales of **CMB polarisation** is recognised as the **next step** in the observation of the Cosmic Microwave Background radiation:

- **LiteBIRD** will be launched at the beginning of 2030s;
- The Simons Observatory (**SO**) is taking data NOW;
- CMB-Stage IV (**S4**) will start at the end of 2020s.



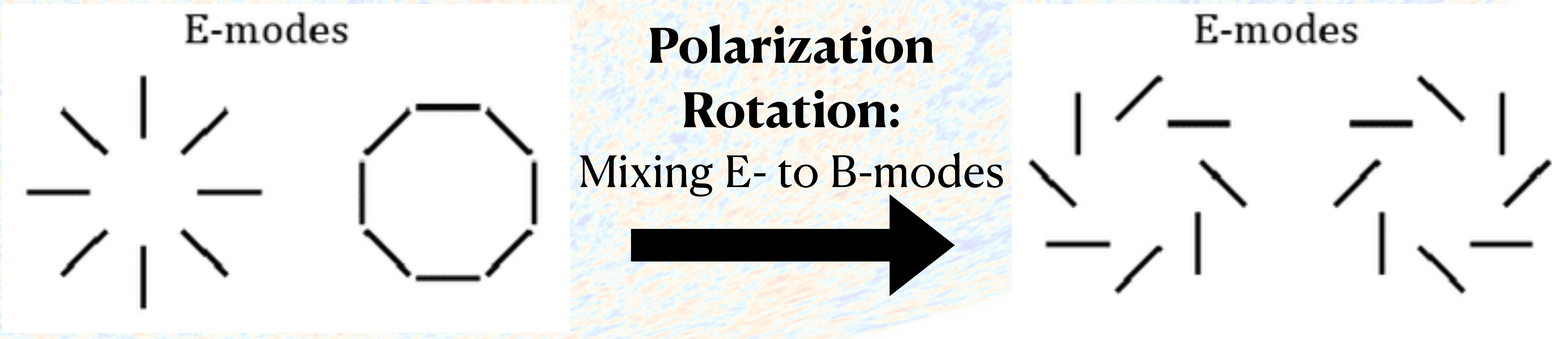
Impact on CMB polarization spectra



$$\alpha = \alpha_0 + \delta\alpha(\vec{n})$$

Isotropic :
non-vanishing EB and TB spectra and the mixing between EE and BB spectra

Anisotropic:
mixes the EE and BB spectra by introducing a coupling among different multipoles



First works

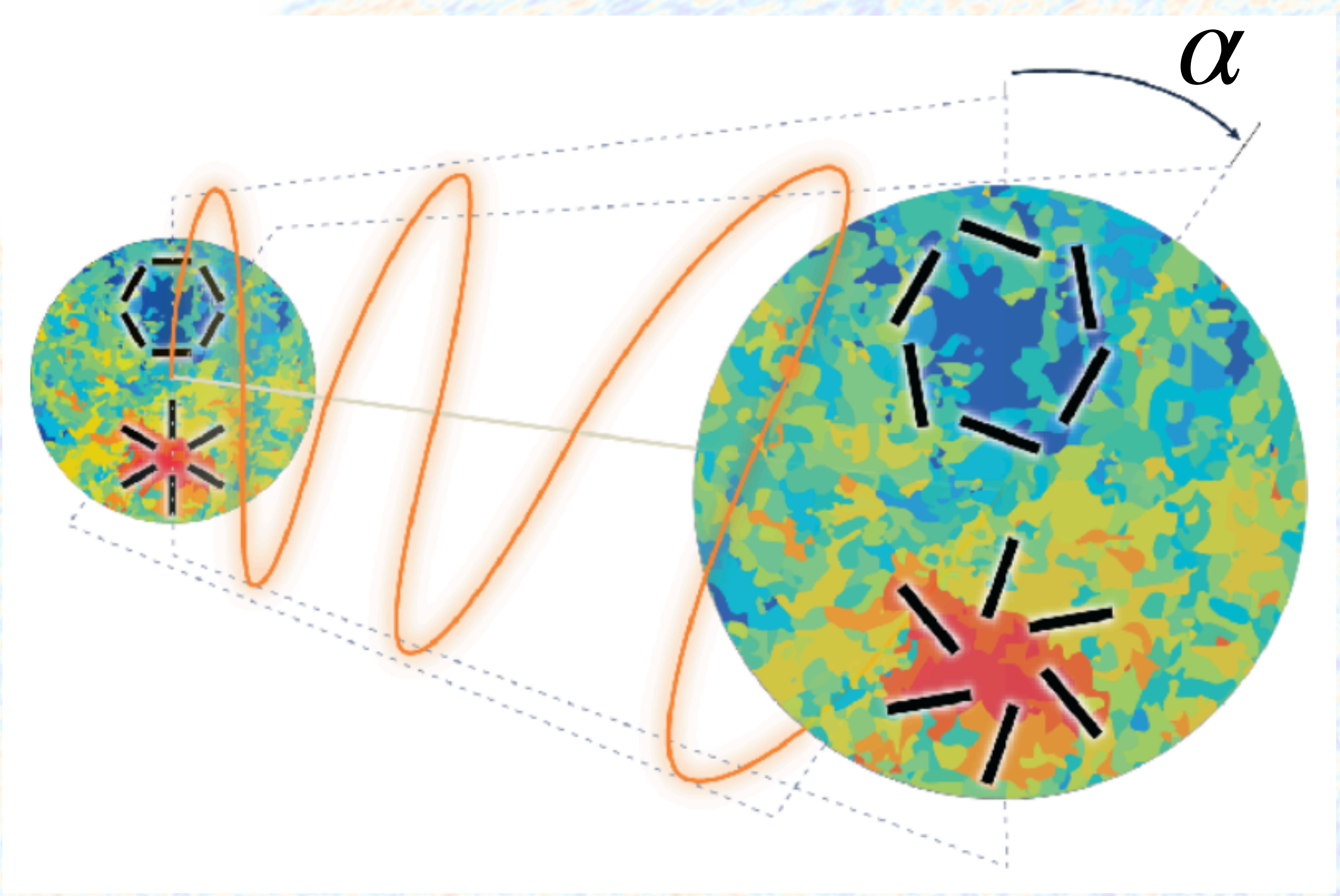
Uniform Rotation

Carroll+(1990,1991) derived constraints on α_0 from 10 **high-redshift radio galaxies**, yielding. $\alpha_0 = -0.6^\circ \pm 1.5^\circ$. Komatsu+ (2008) report limit from the **WMAP5** (combining low-ℓ and high-ℓ dataset) as $\alpha_0 = -1.7^\circ \pm 2.1^\circ$. Xia+(2008) using both **WMAP5 and BOOMERanG 2003** $\alpha_0 = -2.6^\circ \pm 1.9^\circ$. Wu+(2009) measure $\alpha_0 = 0.55^\circ \pm 0.82^\circ(\text{rand}) \pm 0.5^\circ(\text{syst})$ using **QUaD**'s 100 and 150 GHz TB and EB spectra over the multipole range $200 < \ell < 2000$.

Anisotropic Rotation

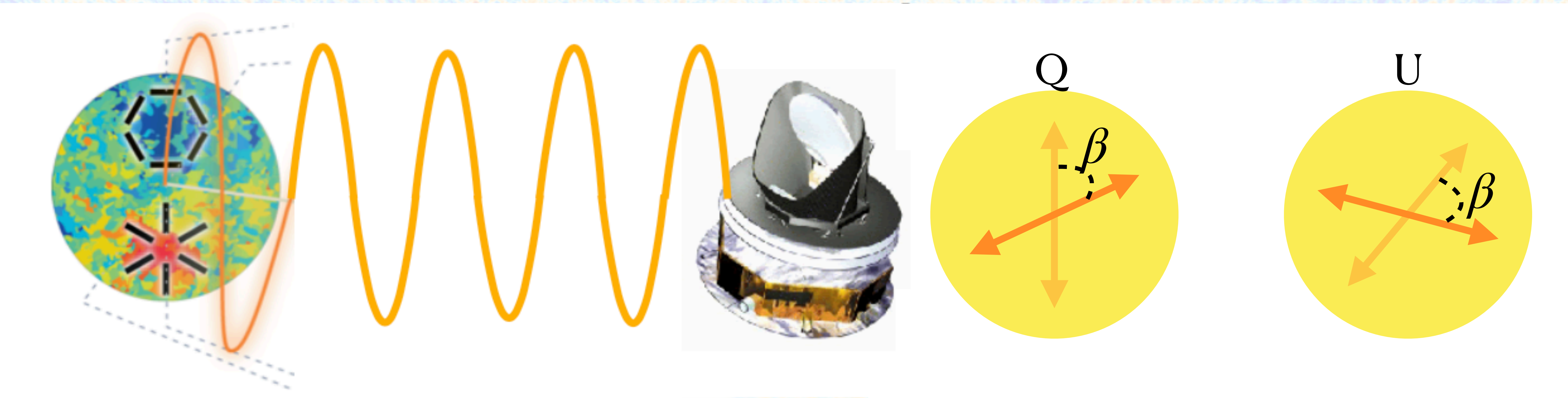
Kamionkowski (2010) sets an upper bound on the variance of $C_\ell^{\alpha\alpha}$ using **AGN**, $< \alpha^2 >^{1/2} \lesssim 3.7^\circ$.
Gluscevic, Hanson, Kamionkowski and Hirata (2012) derived the first CMB constraint on $C_\ell^{\alpha\alpha}$ for multiples between $L = 0$ and $L = 512$ with **WMAP-7**, finding consistency with 0 at each multiple, within 3σ.

Cosmic or Instrumental?



Cosmic birefringence rotates CMB linear polarization plane by α angle

or



Miscalibration of detector's polarization angle β: degenerate with cosmic birefringence angle α

Krachmalnicoff+(2022) - LiteBIRD collab.

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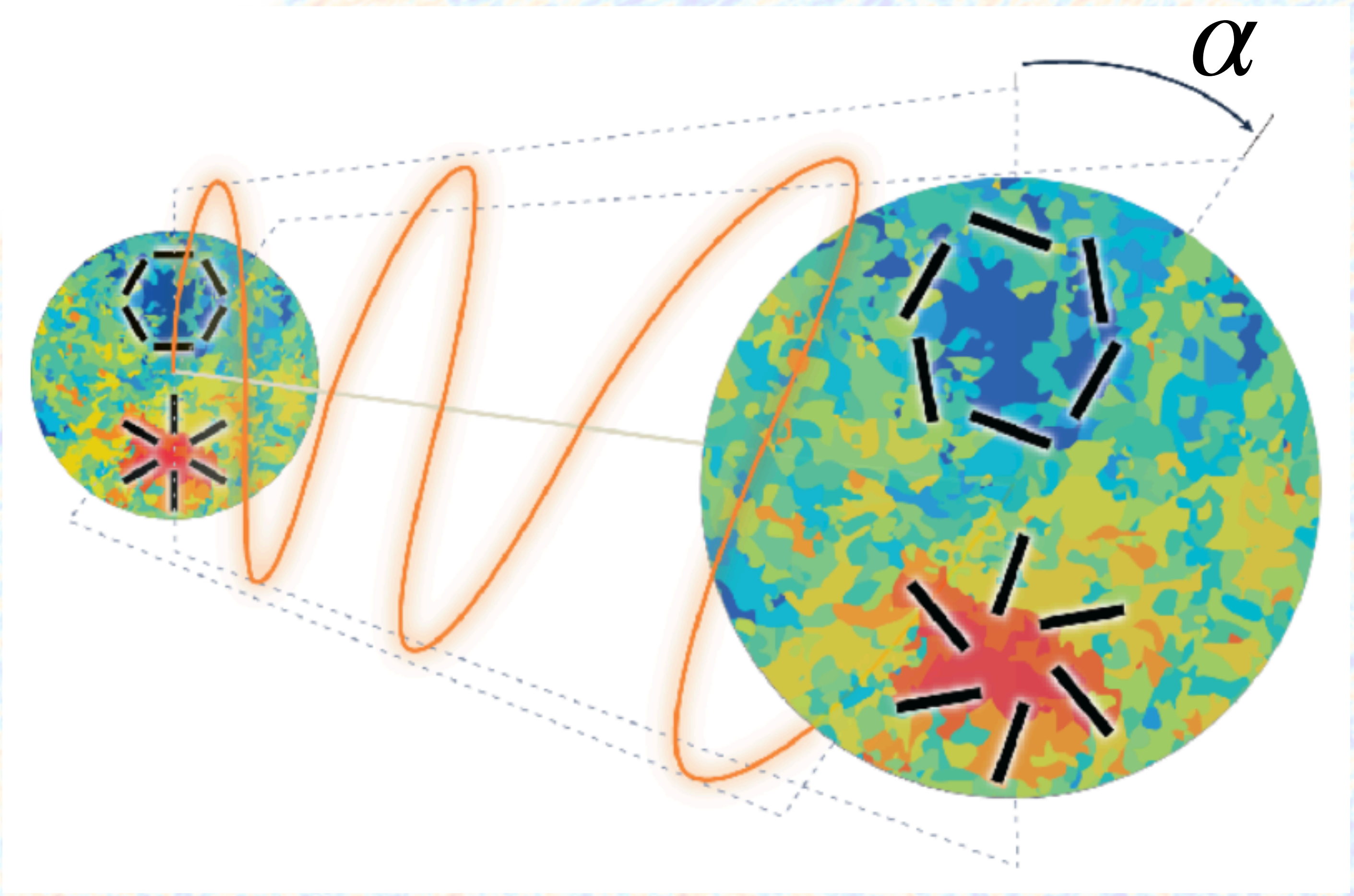


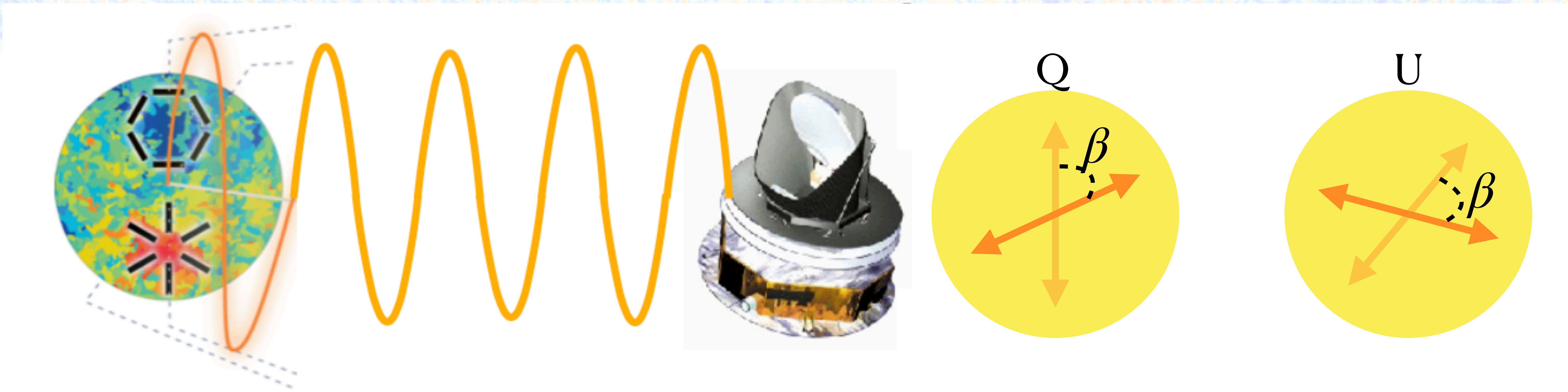
Cosmic birefringence rotates CMB linear polarization plane by α angle

Minami+2019

The sky contains: CMB+Galactic foreground emission. Photons of the **foreground emission do not travel for a long distance**, receiving only a negligible amount of α.

We can **assume** that the **foreground polarization** is **rotated only by** the **miscalibration angle** β.

or



Miscalibration of detector's polarization angle β: degenerate with cosmic birefringence angle α

Krachmalnicoff+(2022) - LiteBIRD collab.

Constraining isotropic birefringence angle

Minami&Komatsu(2020)	Diego-Palazuelos+(2022)	Eskilt&Komatsu(2022)
PR3	**PR4**	***Planck* + WMAP**
$\alpha_0 = (0.35 \pm 0.14)^\circ$	$\alpha_0 = (0.30 \pm 0.11)^\circ$	$\alpha_0 = 0.342^{\circ}{}^{+0.094^\circ}_{-0.091^\circ}$

- Planck (+WMAP) maps (cross-correlatating different frequency maps)
- Applying masks (3 type: bad pixels, bright CO emission, bright point sources)
- Maps to Spectra with NaMaster
- Estimation of β and $\beta + \alpha_0$ maximizing the log-likelihood function

Constraining isotropic birefringence angle

Bortolami, Billi, Gruppuso, Natoli, Pagano (2022)

case	α [deg]
PR3 Commander	0.27 ± 0.05 (stat) ± 0.28 (syst)
PR3 NILC	0.26 ± 0.05 (stat) ± 0.28 (syst)
PR3 SEVEM	0.27 ± 0.05 (stat) ± 0.28 (syst)
PR3 SMICA	0.24 ± 0.05 (stat) ± 0.28 (syst)
NPIPE Commander	0.33 ± 0.04 (stat) ± 0.28 (syst)
NPIPE SEVEM	0.33 ± 0.04 (stat) ± 0.28 (syst)

For reference:

PR2 (SMICA), only EB estimator:

$\alpha_0 = 0.29 \pm 0.05(stat.) \pm 0.28(syst.)$

Planck intermediate results. XLIX. (2016)

Minami&Komatsu(2020) PR3	Diego-Palazuelos+(2022) PR4	Eskilt&Komatsu(2022) *Planck* + WMAP
$\alpha_0 = (0.35 \pm 0.14)^\circ$	$\alpha_0 = (0.30 \pm 0.11)^\circ$	$\alpha_0 = 0.342^{\circ}{}^{+0.094^\circ}_{-0.091^\circ}$

Brief pipeline:

- Cleaned PR3 and NPIPE maps (Commander, NILC, SEVEM and SMICA)
- **Dividing the sky in "patches"**

  ($N_{side} = 8$, $f_{sky,\,patch} \simeq 0.13\,\%$, $N_{tot\,patches} = 768$)
- Applying masks (Galactic foreground and bad pixel)
- SkyPatches to Spectra with NaMaster
- Assuming **isotropic CB in each patches** and applying D^{EB} estimator:

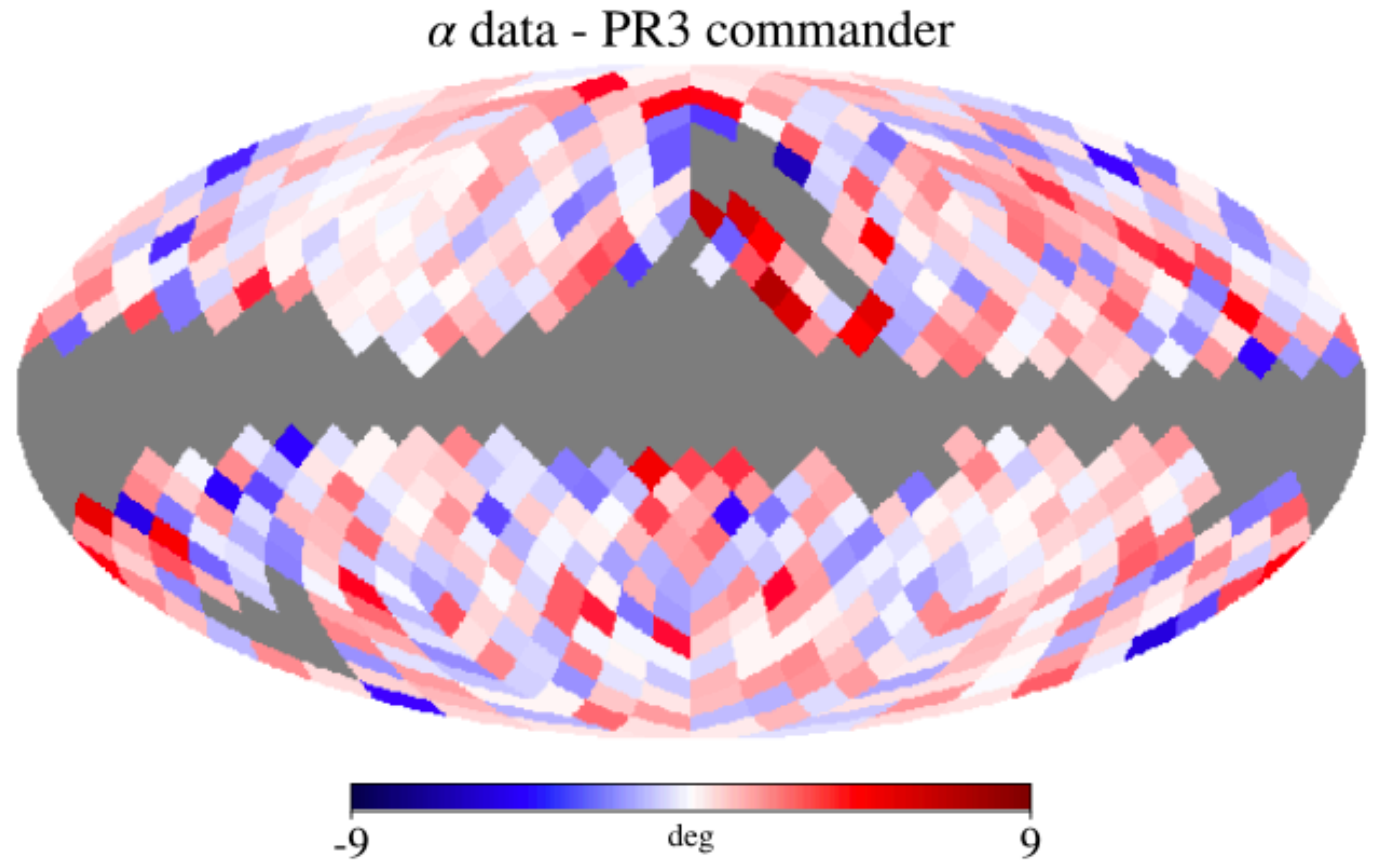
$$D_\ell^{EB}(\alpha) = \hat{C}_\ell^{EB}\cos(4\alpha) - \frac{1}{2}(\hat{C}_\ell^{EE} - \hat{C}_\ell^{BB})\sin(4\alpha)$$

- In each patch, estimation of α_0 maximizing the log-likelihood function —> map of CB angles
- Estimation of α_0 **as monopole of the CB map**

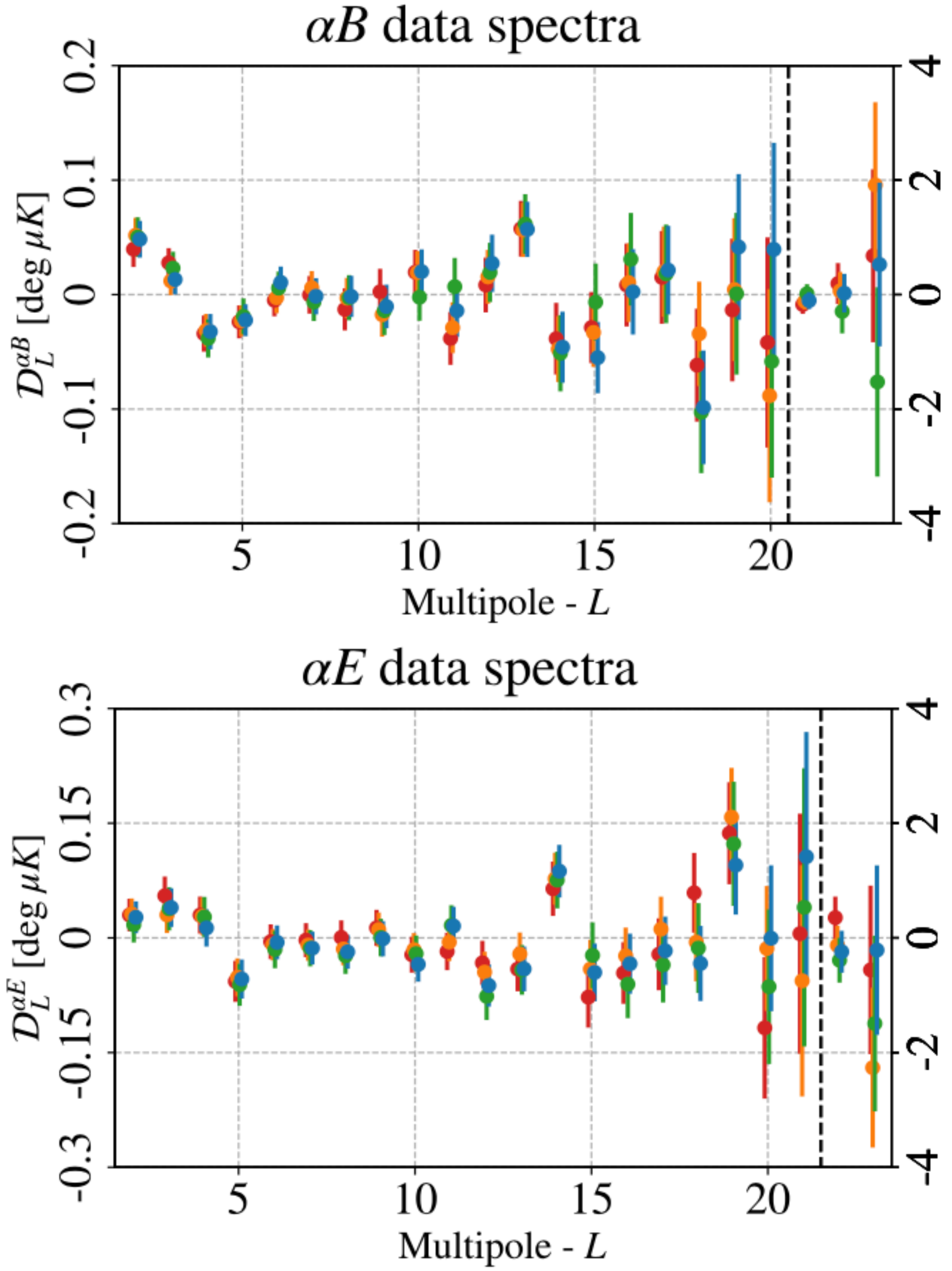
This extends previous work ($N_{side} = 4$) Gruppuso, Molinari, Natoli, Pagano (2020)

Cosmic Birefringence map from PR3

Bortolami, Billi, Gruppuso, Natoli, Pagano (2022)



Cosmic Birefringence angle maps obtained from the PR3 polarization maps for Commander component separation method



Estimates of the cross-correlation spectra $C_\ell^{\alpha E}$ and $C_\ell^{\alpha B}$ (**first time in literature**)

Cosmic Birefringence map from PR3

Constraints from $C_\ell^{\alpha X}$ in terms of the scale-invariant amplitude $A^{\alpha X}$

Bortolami, Billi, Gruppuso, Natoli, Pagano (2022) ($L = 24$)

parameter	Commander	NILC	SEVEM	SMICA
$A^{\alpha\alpha}$ [deg^2] PR3	< 0.007	< 0.007	< 0.010	< 0.007
$A^{\alpha\alpha}$ [deg^2] NPIPE	< 0.010	-	< 0.009	-
$A^{\alpha T}$ [μK deg] PR3	-1.827 ± 0.953	-1.229 ± 0.873	-2.037 ± 1.038	-1.916 ± 0.945
$A^{\alpha E}$ [nK deg] PR3	-3.5 ± 6.0	-1.0 ± 5.6	-9.7 ± 6.0	-7.8 ± 5.6
$A^{\alpha B}$ [nK deg] PR3	2.4 ± 4.0	-1.2 ± 3.7	4.0 ± 4.4	0.3 ± 4.0

$A^{\alpha\alpha}_{\texttt{SMICA}} < 0.104\,\text{deg}^2$ at 95%C.L. ($L = 12$) Gruppuso, Molinari, Natoli, Pagano (2020)

Other constraints on the scale-invariant amplitude $A^{\alpha X}$:

Contreras,Boubel,Scott(2018) - Planck data $\leq 0.018\,\text{deg}^2$

Bianchini, SPT collaboration (2020) $\leq 0.033\,\text{deg}^2$

Namikawa, ACT collaboration (2020)

Cosmic Birefringence map from PR4

$D^{EB}(\alpha)$ estimator —> α angle

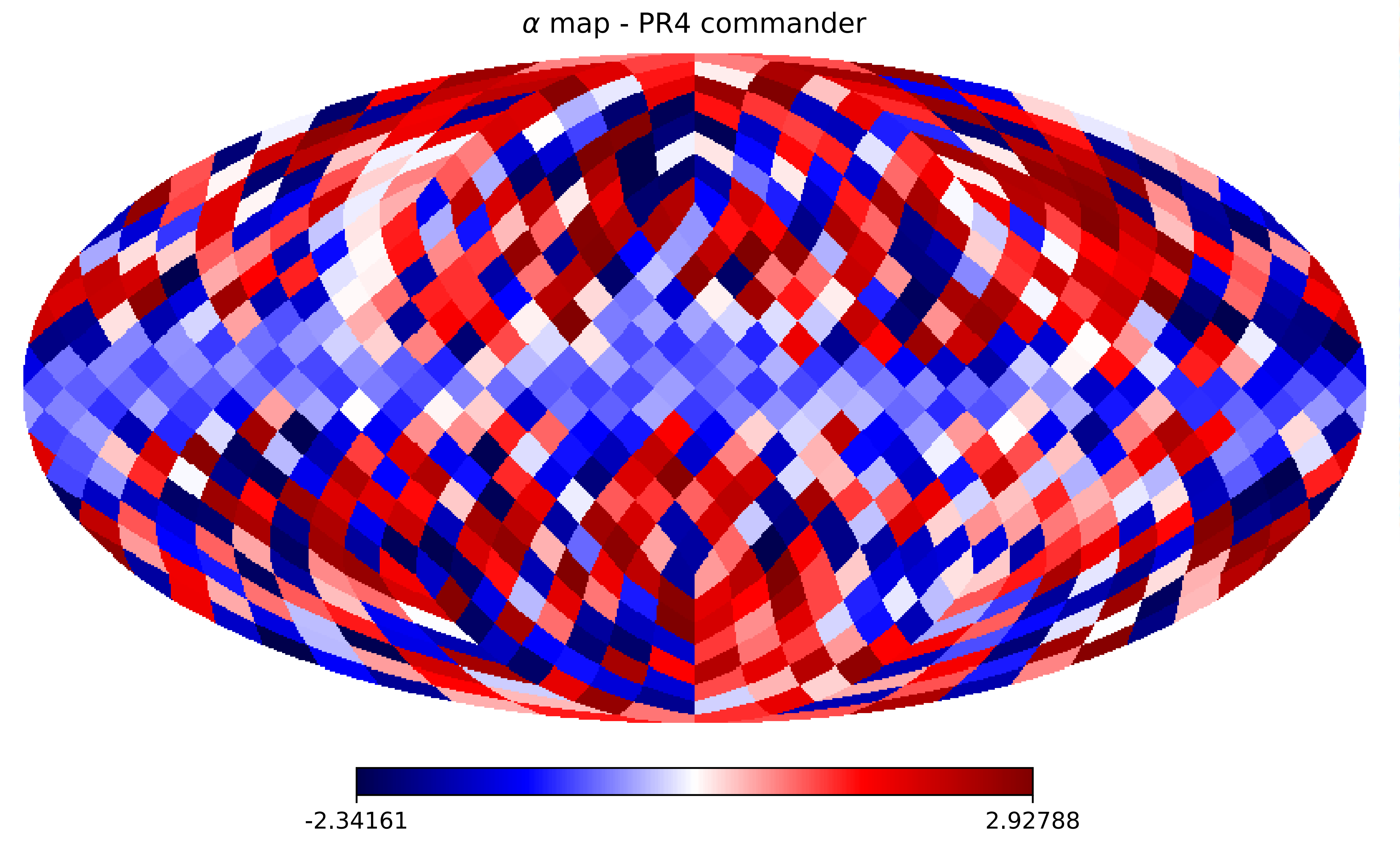
Bortolami, Billi, Gruppuso, Natoli, Pagano (2022)
Gruppuso, Molinari, Natoli, Pagano (2020)

Kind of estimator as in Gluscevic, Hanson, Kamionkowski and Hirata (2012)

$\hat{\alpha}^{EB}_{LM}$ estimator —> α_{LM}

Zagatti, Bortolami, Gruppuso, Natoli, Pagano (TBS)

- L up to $2\,\ell_{max}$ of CMB maps
- Computationally less expensive
- CB spectrum compatible with 0 at $\sim 2\sigma$ (w/o assuming a scale invariant spectrum)



Birefringent Cross-Bispectra

Greco,Bartolo,Gruppuso (2022)

compute the three-point correlation functions of anisotropic birefringence with the "observed" CMB fields

observed means accounting from weak gravitational lensing and eventual birefringence effects

$$\langle a_{X,\ell_1 m_1} a_{Y,\ell_2 m_2} a_{Z,\ell_3 m_3} \rangle = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B^{XYZ}_{\ell_1 \ell_2 \ell_3} \quad X = \alpha, \quad Y, Z = \alpha, T, E, B$$

New cosmological observables that:

- carry signatures of parity-breaking physics
- survive in a regime of purely anisotropic cosmic birefringence
- $\neq 0$ even if the fields involved are Gaussian

SNR for birefringent bispectra accounting for LiteBIRD-like instrumental noise in a pure anisotropic regime ($D^{\alpha\alpha}_\ell \simeq 0.1\ \text{deg}^2$)

Bispectrum	SNR
$\delta\alpha\,TE$	≈ 0.0661
$\delta\alpha\,TB$	≈ 4.0635
$\delta\alpha\,EB$	≈ 7.5658
$\delta\alpha\,EE$	≈ 0.0543
$\delta\alpha\,BB$	≈ 0.0004

Beyond Cosmic Birefringence

We consider the ***minimal Standard Model Extension*** - contains only renormalizable operators with mass dimension ≤ 4

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}A_\beta(k_{AF})_\alpha F_{\mu\nu} - \frac{1}{4}(k_F)^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}\right]$$

Standard Maxwell term

CPT-odd term

The time and the space components of the coupling k_{AF} lead to isotropic and anisotropic birefringence, respectively

CPT-even term

The couplings k_F lead to a conversion of linear polarization (EE and BB spectra) into circular polarization (VV spectrum)

Caloni, Giardiello, **Lembo**, Gerbino, Gubitosi, Lattanzi, Pagano (2023)

Beyond Cosmic Birefringence

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applying the *dark crystal formalism* as described in

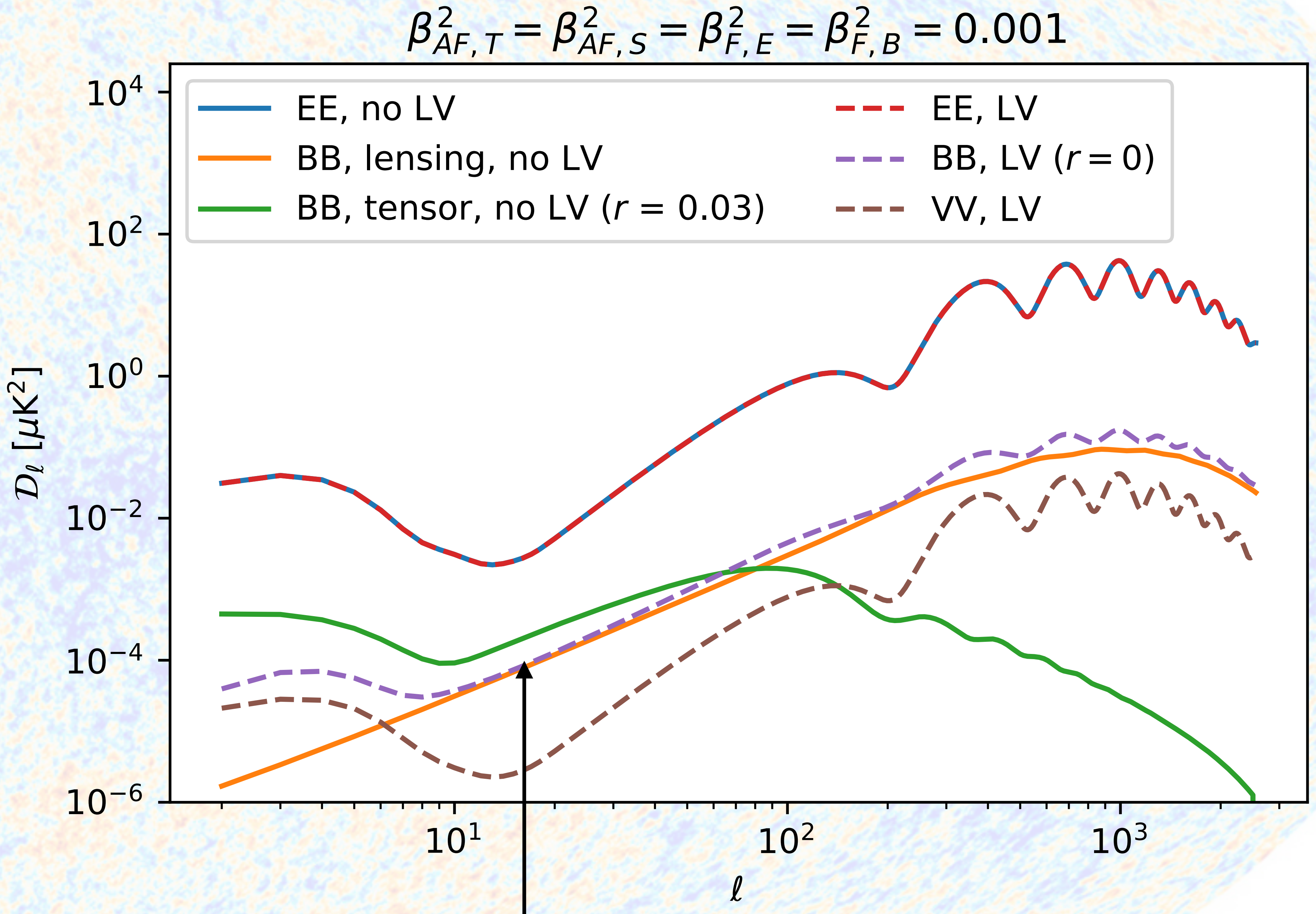
Lembo, Lattanzi, Pagano, Gruppuso, Natoli, Forastieri (PRL2021)

CMB spectra (including EB, TB, VV ≠ 0) as function of some effective parameters:

$\beta^2_{AF,\,T/S}$ related to time/space components of k_{AF} **(CPT-odd)**

β^2_F depends of the components of k_F in a non-trivial way **(CPT-even)**

The spectra are generated using camb-cpt (our custom-made version of camb)



Spurious B-mode component even in absence of tensor modes (dashed purple line).

Caloni, Giardiello, **Lembo**, Gerbino, Gubitosi, Lattanzi, Pagano (2023)

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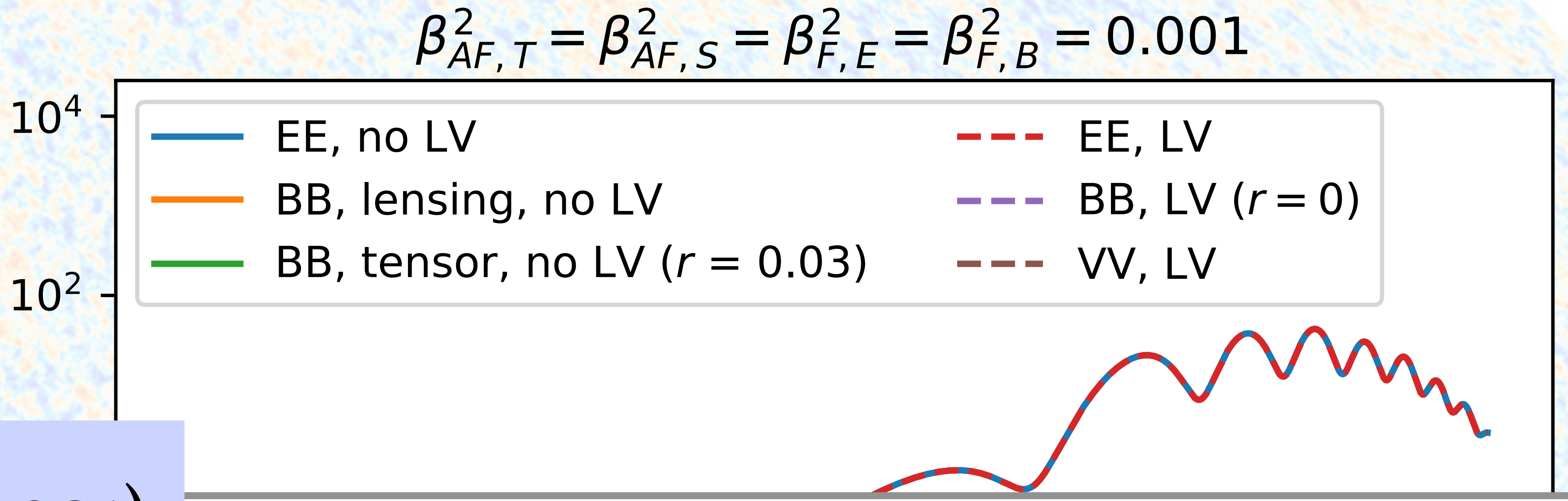

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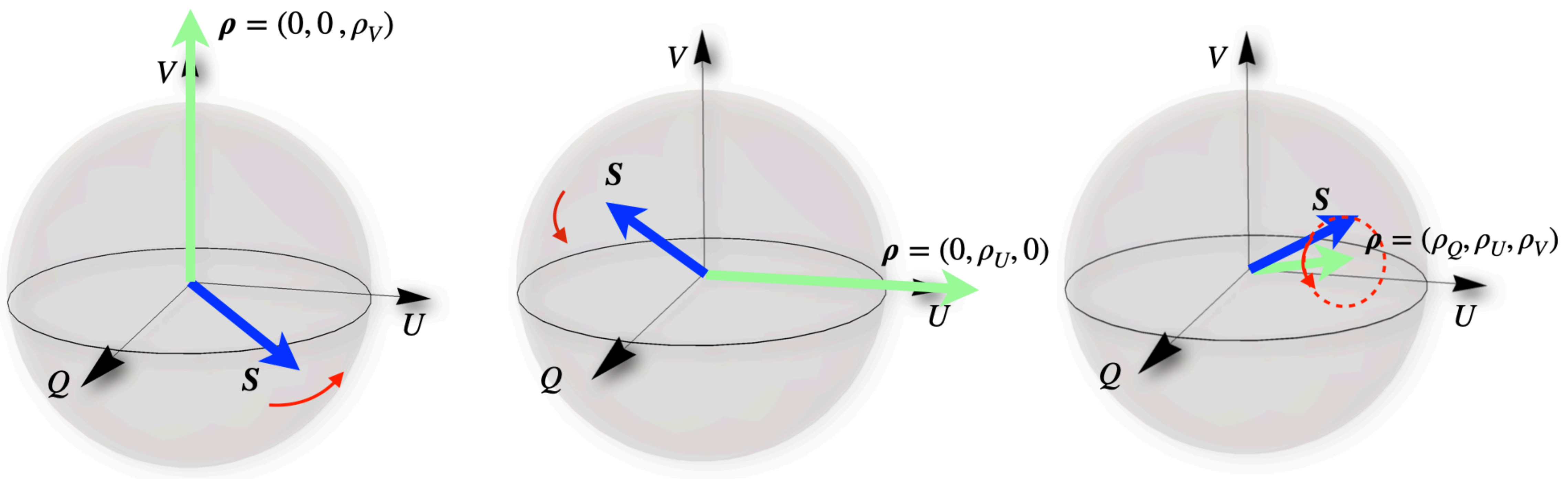
The idea:

CMB photons propagate through the Universe. **We *interprete* our Universe as a medium with an "effective" susceptibility tensor.** If this susceptibility tensor is anisotropic and/or parity-violating, mixing among polarization component of CMB radiation occurred. We can then **link the components of the susceptibility tensor to phenomenological parameters describing this conversion among CMB polarization states.**



$$\frac{\mathrm{d}}{\mathrm{d}s} \mathbf{S} = \boldsymbol{\rho} \times \mathbf{S}$$

where

$$\mathbf{S} = (Q, U, V)$$

$$\boldsymbol{\rho} = (\rho_Q, \rho_U, \rho_V)$$

Beyond Cosmic Birefringence

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}A_\beta(k_{AF})_\alpha F_{\mu\nu} - \frac{1}{4}(k_F)^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}\right]$$

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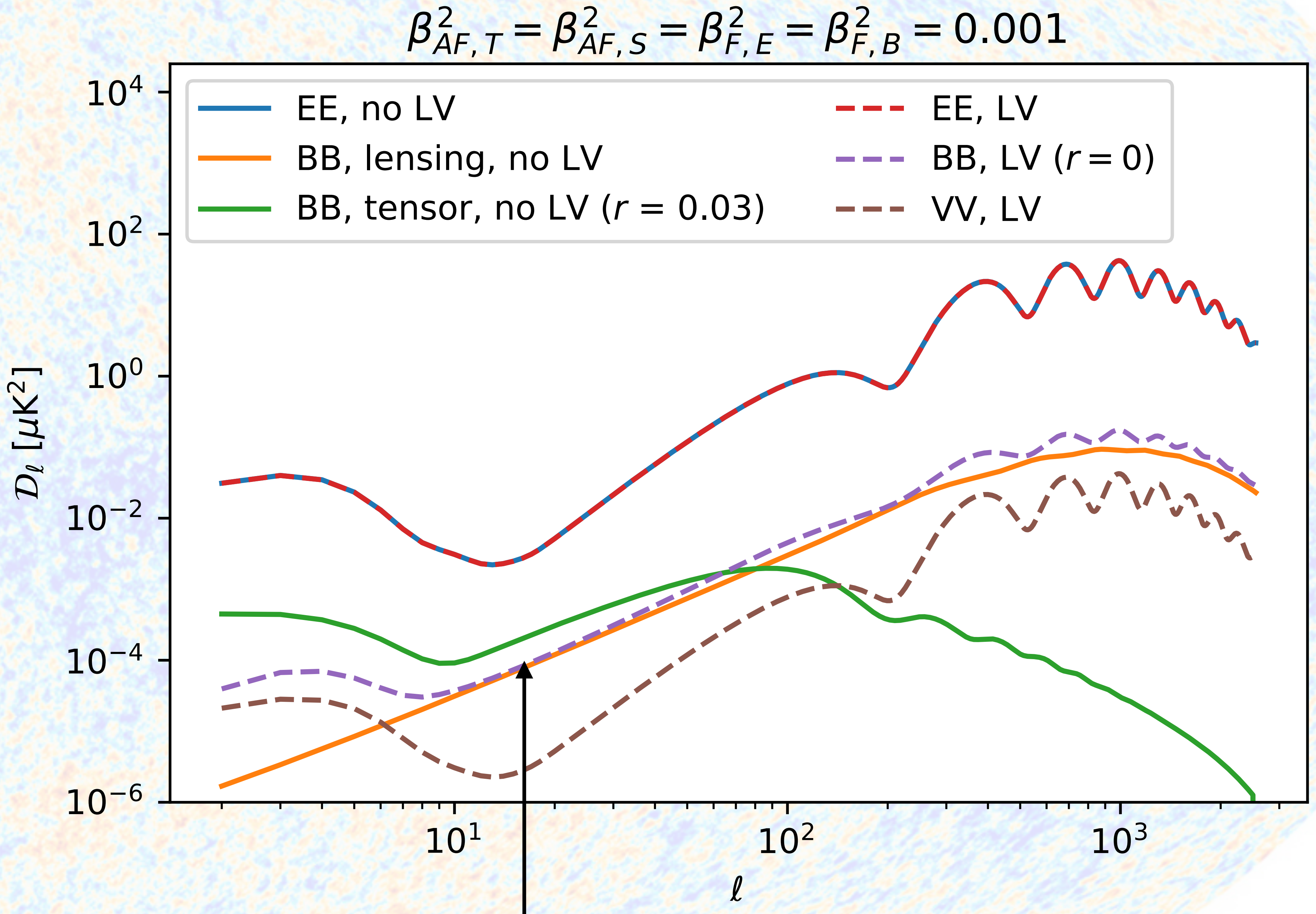

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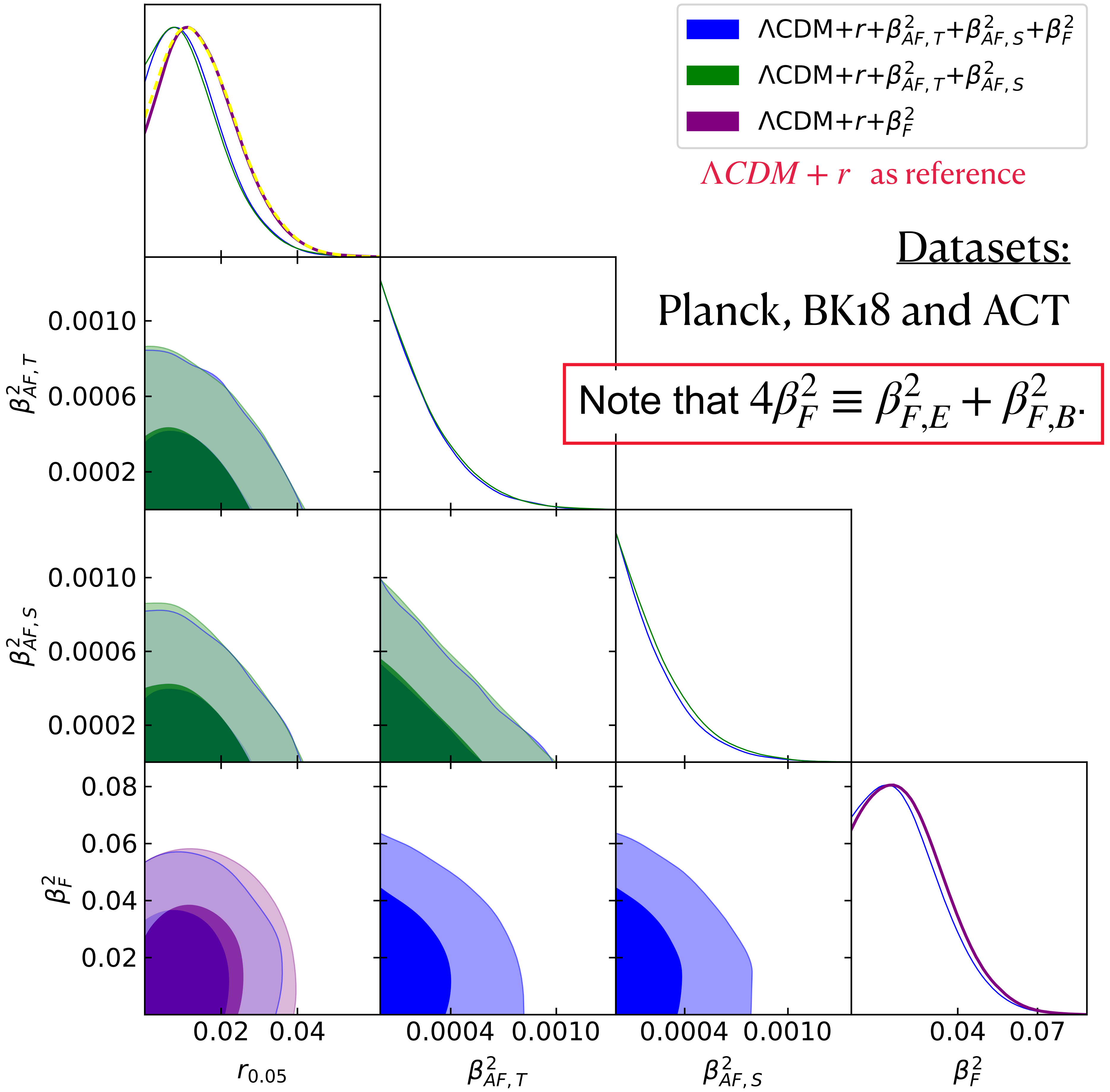

$\Lambda CDM + r$ as reference

Datasets:

Planck, BK18 and ACT

Note that $4\beta^2_F \equiv \beta^2_{F,E} + \beta^2_{F,B}$.

Comparison with previous works

First comprehensive study of the signatures of Lorentz violation in electrodynamics on CMB anisotropies.

(see V.A. Kostelecky and N. Russell, Data Tables for Lorentz and CPT Violation arXiv:0801.0287)

CPT-odd

Our bounds are the strongest to date, both considering CMB and other sources.

CPT-even

Our bound improves previous constraints by roughly one order of magnitude. This bound is only overcome by those obtained from optical polarimetry of extragalactic sources.

Caloni, Giardiello, **Lembo**, Gerbino, Gubitosi, Lattanzi, Pagano (2023)

Summary and future prospects

- **Scientists within the COSMOS network have significantly advanced our understanding of cosmic birefringence**, by introducing novel methodologies and exploring Planck data to establish more rigorous constraints on both the isotropic birefringence angle and the scale-invariant amplitude of the anisotropic birefringence spectrum.
- Minami&Komatsu (2020), Diego-Palazuelos+(2022) and Eskilt&Komatsu (2022) have suggested **an hint of detection of isotopic birefringence**, excluding $\alpha_0 = 0$ with a significance ranging from 2.4σ to 3.6σ. This motivates further investigations and suggests us to improve our knowledge of Galactic foregrounds.
- Our classical theory of gravitation **fails to be predictive** in physically relevant regimes. If Quantum Gravity is the answer we can only look for low energy **"relic signatures"**, such as violation of symmetries (Parity, Lorentz).
- **Constraining Cosmology through CMB polarization is the focus of next-decade CMB experiments.** Forthcoming CMB experiments will reach an **unprecedented sensitivity to linear CMB polarization.** B-modes are the key observable of the coming decade. A parity-violating term, as extension of the standard model, generates spurious B-mode component that acts as a **potential contaminant for all the measurements of primordial B-modes.**

Summ

- **Scientists withi** ... **tanding of cosmic birefringence**, b ... ablish more rigorous constraints on b ... e of the anisotropic birefringence spe
- Minami&Komatsu ... ggested **an hint of detection of iso** ... n 2.4σ to 3.6σ. This motivates further ... grounds.
- Our classical theo ... uantum Gravity is the answer we can on ... (Parity, Lorentz).
- **Constraining C** ... **CMB experiments.** Forthcoming CM ... **olarization.** B-modes are the key obse ... the standard model, generates spuriou ... **e measurements of primordial B-mo**

Thanks for your attention!

Backslides

Impact on CMB polarization spectra

$$\alpha = \alpha_0 + \delta\alpha(\vec{n})$$

Isotropic (α_0) — Anisotropic ($\delta\alpha(\vec{n})$)

$$L = \ell + \ell_1 + \ell_3$$

$$M_{\ell\ell_1\ell_3} = \frac{(2\ell_1+1)(2\ell_3+1)}{\pi}\begin{pmatrix} \ell & \ell_1 & \ell_3 \\ 2 & -2 & 0 \end{pmatrix}^2$$

$$4\pi V_\alpha = \sum_\ell (2\ell+1)\; C_\ell^{\alpha\alpha}$$

$$C_\ell^{TT} = \tilde{C}_\ell^{TT}$$

$$C_\ell^{TE} = \tilde{C}_\ell^{TE}\cos(2\alpha_0)\left(1-2V_\alpha\right)$$

$$C_\ell^{TB} = \tilde{C}_\ell^{TE}\sin(2\alpha_0)\left(1-2V_\alpha\right)$$

$$C_\ell^{EE} = \left[\tilde{C}_\ell^{EE}\cos^2(2\alpha_0)+\tilde{C}_\ell^{BB}\sin^2(2\alpha_0)\right]\left(1-4V_\alpha\right) + \sum_{\ell_1\ell_3}\left[\left(1-(-1)^L\cos(4\alpha_0)\right)\tilde{C}_{\ell_1}^{EE} + \left(1+(-1)^L\cos(4\alpha_0)\right)\tilde{C}_{\ell_1}^{BB}\right] C_{\ell_3}^{\alpha\alpha}\,\frac{M_{\ell\ell_1\ell_3}}{2}$$

$$C_\ell^{BB} = \left[\tilde{C}_\ell^{BB}\cos^2(2\alpha_0)+\tilde{C}_\ell^{EE}\sin^2(2\alpha_0)\right]\left(1-4V_\alpha\right) + \sum_{\ell_1\ell_3}\left[\left(1-(-1)^L\cos(4\alpha_0)\right)\tilde{C}_{\ell_1}^{BB} + \left(1+(-1)^L\cos(4\alpha_0)\right)\tilde{C}_{\ell_1}^{EE}\right] C_{\ell_3}^{\alpha\alpha}\,\frac{M_{\ell\ell_1\ell_3}}{2}$$

$$C_\ell^{EB} = \sin(4\alpha_0)\,\frac{1}{2}\left(\tilde{C}_\ell^{EE}-\tilde{C}_\ell^{BB}\right)\left(1-4V_\alpha\right) + \sin(4\alpha_0)\sum_{\ell_1\ell_3}\left[\frac{1}{2}\left(\tilde{C}_{\ell_1}^{EE}-\tilde{C}_{\ell_1}^{BB}\right)\right] C_{\ell_3}^{\alpha\alpha}\,(-1)^{L+1}M_{\ell\ell_1\ell_3}$$

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